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Thermo-acoustical waves in linear thermo-elastic materials

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SUMMARY
Coupled waves of thermal and mechanical jumps in linear thermo-elastic materials are analysed. General linear anisotropic constitutive equations of thermo-elastic materials are derived from the Clausius–Duhem inequality and Vernotte's heat conduction law is adopted. The waves are defined to have jumps in acceleration and in temperature rate and the four-dimensional thermo-acoustical propagation condition is obtained. The differential equations which govern the variation of the wave amplitudes are obtained. For waves in linear isotropic thermo-elastic materials, there are four principal waves. Two shear waves are purely mechanical and propagate with constant amplitude, while two thermo-longitudinal waves have different propagation velocities: one is larger and other smaller than the purely mechanical longitudinal wave velocity, and their amplitudes decay, in general, exponentially in time.

1. Introduction

In the classical theory *Fourier's law*:

$$q_i = -\varkappa T_{,i} \tag{1.1}$$

is assumed as a law of heat conduction, where q_i is the heat flux, T is the temperature and $\varkappa$ is a positive constant called the conductivity. Here and henceforth a comma followed by a suffix denotes the partial derivative with respect to a coordinate. As a consequence of (1.1) the temperature field in a material is governed by a parabolic equation, which means that a thermal disturbance must propagate with infinite velocity. This fact shows that Fourier's law is valid only for a slowly varying phenomenon or a quasi-equilibrium state.

In order to remedy this unpleasant feature Vernotte [1] proposed a modified Fourier's law:

$$\dot{q}_i = -\frac{1}{\tau}(q_i + \varkappa T_{,i}), \tag{1.2}$$

where τ denotes the relaxation time. Law (1.2) means that the dynamic heat conduction relaxes with time τ and it is reduced to Fourier's law (1.1) in an equilibrium state.

Applying (1.2) to the coupled wave propagation of thermo-elasticity, Lord and Shulman [2], Popov [3] and Achenbach [4] discussed *one*-dimensional wave propagation in isotropic linear elastic materials with thermal influence. Gurtin and Pipkin [5] investigated the *temperature rate wave* in a rigid material with memory, and Chen [6, 7] and McCarthy [8] analysed its growth and decay.

In this paper *three-dimensional thermo-acoustical waves* are discussed theoretically. In Sect. 2 the constitutive equations of an anisotropic linear thermo-elastic material are defined from the *Clausius–Duhem inequality* and the anisotropic heat conduction law is defined. In Sects. 3 and 4 the properties of thermo-acoustical waves in anisotropic thermo-elastic materials are discussed generally and in the last section waves for isotropic cases are discussed.

2. Basic equations

A thermo-elastic material is defined by the constitutive equations:

$$\psi = \hat{\psi}(\varepsilon_{ij}, \theta), \quad \sigma_{ij} = \hat{\sigma}_{ij}(\varepsilon_{ij}, \theta), \quad \eta = \hat{\eta}(\varepsilon_{ij}, \theta), \tag{2.1a, b, c}$$

where ψ, σ_{ij} and η denote, respectively, the specific Helmholtz free energy, the stress tensor and the specific entropy, while ε_{ij} is the strain tensor and

$$\theta \equiv (T - T_0)/T_0 \tag{2.2}$$

denotes the dimensionless temperature. Here and henceforth the suffix zero denotes a quantity evaluated in an equilibrium static state.

Coleman and Noll [9] introduced the Clausius–Duhem inequality:

$$\rho(\dot{\psi} + \eta \dot{T}) - \sigma_{ij} d_{ij} + \frac{1}{T} q_i T_{,i} \leqq 0\,, \tag{2.3}$$

which must be satisfied for all admissible processes, where ρ and $d_{ij} \equiv (\frac{1}{2})(v_{i,j} + v_{j,i})$ denote, respectively, the density and the deformation rate tensor, v_i is the velocity of a material particle, and the summation convention with repeated suffixes is applied. From (2.1a) and (2.3) we can conclude that

$$\sigma_{ij} = \rho \frac{\partial \hat{\psi}}{\partial \varepsilon_{ij}}\,, \quad \eta = -\frac{1}{T} \frac{\partial \hat{\psi}}{\partial \theta}\,, \tag{2.4a, b}$$

$$q_i T_{,i} \leqq 0\,. \tag{2.5}$$

Now we assume that the concerned deformation and temperature deviation from an equilibrium state are so small that the free energy may be approximated by the quadratic form:

$$\psi = \tfrac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + C_{ij} \varepsilon_{ij} \theta + \tfrac{1}{2} C \theta^2\,, \tag{2.6}$$

where C_{ijkl}, C_{ij} and C are material constants and have the following symmetry relations:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}\,, \qquad C_{ij} = C_{ji}\,. \tag{2.7a, b}$$

Substituting (2.6) into (2.4) and restricting the relations to be linear we have

$$\sigma_{ij} = \rho_0 (C_{ijkl} \varepsilon_{kl} + C_{ij} \theta)\,, \quad \eta = -\frac{1}{T_0} (C_{ij} \varepsilon_{ij} + C\theta)\,. \tag{2.8a, b}$$

The law of balance of energy is expressed by

$$\rho\,(\psi + \eta T)^{\cdot} = \sigma_{ij} d_{ij} - q_{i,i} \tag{2.9}$$

with the assumption of no heat supply. Applying (2.4) to (2.9) we have

$$\dot{\eta} = -q_{i,i}/(\rho_0 T_0)\,. \tag{2.10}$$

From (2.8) and (2.10) we have

$$\dot{\sigma}_{ij} = \rho_0 (C_{ijkl} d_{kl} + C_{ij} \dot{\theta})\,, \tag{2.11}$$

$$q_{i,i} = \rho_0 (C_{ij} d_{ij} + C \dot{\theta})\,. \tag{2.12}$$

Any material particle must satisfy the law of balance of linear momentum:

$$\sigma_{ij,j} = \rho_0 \dot{v}_i\,, \tag{2.13}$$

where the body force is neglected.

Now we generalize Vernotte's isotropic heat conduction law (1.2) to an anisotropic law, that is, we assume

$$\dot{q}_i = -\nu_{ij} (q_j + K_{jk} \theta_{,k})\,, \tag{2.14}$$

where ν_{ij} and $K_{ij} \equiv \varkappa_{ij} T_0$ denote, respectively, the inverse relaxation time tensor and the conductivity tensor.

Relations (2.11)–(2.14) constitute thirteen equations with thirteen variables v_i, θ, σ_{ij} and q_i.

3. Velocities and amplitudes of thermo-acoustical waves

A singular surface associated with the velocity and the temperature is called a *thermo-acoustical wave*★ if the following two conditions hold:

(*i*) *v_i and θ are continuous everywhere.*

(*ii*) *$\dot{v}_i$, $v_{i,j}$, $\ddot{v}_i$, $\dot{v}_{i,j}$, $v_{i,jk}$, $\dot{\theta}$, $\theta_{,i}$, $\ddot{\theta}$, $\dot{\theta}_{,i}$, and $\theta_{,ij}$ have jump discontinuities across the singular surface but are continuous everywhere else.*

Also we assume that σ_{ij} and q_i are continuous across the singular surface while their first- and second-order derivatives may have finite jump discontinuities.

The geometrical and kinematical compatibility conditions of any quantity f, with the assumption $[f]=0$ across a singular surface, are given by

$$[f_{,i}]=\bar{f}n_i\,,\quad [\dot{f}]=-U\bar{f}\,, \tag{3.1}$$

where n_i and U denote, respectively, the unit normal and propagation normal velocity of the surface and

$$\bar{f}\equiv[f_{,i}]\,n_i\,, \tag{3.2}$$

where a square bracket denotes a difference of two values of a quantity adjacent to the both sides of the surface [10, Eqs. (180.5)].

Applying the compatibility conditions (3.1) to (2.11)–(2.14) we have

$$-U\bar{\sigma}_{ij}=\rho_0(C_{ijkl}n_l\bar{v}_k-UC_{ij}\bar{\theta})\,, \tag{3.3a}$$

$$\bar{q}_in_i=\rho_0(C_{ij}n_j\bar{v}_i-UC\bar{\theta})\,, \tag{3.3b}$$

$$\bar{\sigma}_{ij}n_j=-\rho_0U\bar{v}_i\,, \tag{3.3c}$$

$$U\bar{q}_i=v_{ij}K_{jk}n_k\bar{\theta}\,. \tag{3.3d}$$

Eliminating $\bar{\sigma}_{ij}$ and $\bar{q}_i$ from (3.3) we have the propagation conditions of the thermo-acoustical wave:

$$(Q_{ik}-U^2\delta_{ik})\bar{v}_k-UQ_i\bar{\theta}=0\,, \tag{3.4a}$$

$$-UQ_i\bar{v}_i+(Q+U^2C)\bar{\theta}=0\,, \tag{3.4b}$$

where

$$Q_{ik}\equiv C_{ijkl}n_jn_l\,,\quad Q_i\equiv C_{ij}n_j\,,\quad Q\equiv\frac{n_iv_{ij}K_{jk}n_k}{\rho_0} \tag{3.5}$$

and Q_{ik} is called the *acoustical tensor* of the purely mechanical acceleration wave. The vector $\bar{v}_i$ and the scalar $\bar{\theta}$ denote, respectively, the mechanical and the thermal components of the thermo-acoustical wave.

Now we introduce a four-dimensional inner-product space $\mathscr{V}_4$, which consists of a physical three-dimensional space and a one-dimensional real space. The amplitudes $\bar{v}_i$ and $\bar{\theta}$ are combined and represented by a single vector a_α in $\mathscr{V}_4$, where the greek suffix runs from one to four and $a_i=\bar{v}_i$ and $a_4=\bar{\theta}$. Then (3.4) are expressed by

$$R_{\alpha\beta}a_\beta=0\,, \tag{3.6}$$

where

$$\|R_{\alpha\beta}\|\equiv\begin{Vmatrix}Q_{ik}-U^2\delta_{ik} & -UQ_i\\ -UQ_i & Q+U^2C\end{Vmatrix} \tag{3.7}$$

is called the *thermo-acoustical tensor.*

★ A purely mechanical wave, which is defined by a non-vanishing jump of acceleration is called generally an *acceleration wave*, while a purely thermal wave, which is defined by a non-vanishing jump of temperature rate is called a *temperature rate wave* [5].

The propagation velocities are obtained as solutions of

$$\det (R_{\alpha\beta}) = 0 \tag{3.8}$$

and we have, in general, four principal waves whose propagation velocities are solutions of (3.8) and whose amplitudes have directions parallel to the principal axes of $R_{\alpha\beta}$.

When the material has no thermo-mechanical coupling, $C_{ij}=0$ from (2.8). Then from (3.7) the waves are separated into purely mechanical and purely thermal waves and their velocities are given, respectively, by

$$\det (Q_{ik} - U^2 \delta_{ik}) = 0\,, \quad U^2 = -Q/C\,. \tag{3.9}$$

Furthermore, if the material is a non-conductor, we have $C=0$, then $a_4=0$, which indicates that there is no temperature rate wave.

4. Variation of amplitudes of plane thermo-acoustical waves

Thomas' iterated compatibility conditions of second-order are given by

$$[f_{,ij}] = \bar{\bar{f}} n_i n_j\,, \quad [\dot{f}_{,i}] = (-U\bar{\bar{f}} + \dot{\bar{f}}) n_k\,, \quad [\ddot{f}] = U^2 \bar{\bar{f}} - 2U\dot{\bar{f}}\,, \tag{4.1a, b, c}$$

where

$$\bar{\bar{f}} = [f_{,ij}] n_i n_j \tag{4.2}$$

and where any quantity differentiated with respect to the coordinates on the plane wave front may be assumed to vanish. (Cf. [11] and [10, Eq. (176.8) and Eqs. (181.8)]).

Differentiating (2.11)–(2.14) with respect to time and applying (4.1) to them and eliminating $\bar{\bar{\sigma}}_{ij}$, $\bar{\bar{q}}_i$, $\bar{\sigma}_{ij}$ and $\bar{q}_i$, we have

$$2U\dot{\bar{v}}_i + Q_i \dot{\bar{\theta}} = -(Q_{ik} - U^2 \delta_{ik}) \bar{\bar{v}}_k + U Q_i \bar{\bar{\theta}}\,, \tag{4.3a}$$

$$Q_i \dot{\bar{v}}_i - 2UC\dot{\bar{\theta}} + (P/U)\bar{\theta} = U Q_i \bar{\bar{v}}_i - (Q + U^2 C)\bar{\bar{\theta}}\,, \tag{4.3b}$$

where

$$P \equiv (1/\rho_0) n_i v_{ij} v_{jk} K_{km} n_m\,. \tag{4.4}$$

The $\mathscr{V}_4$-space representation of (4.3) is given by

$$S_{\alpha\beta} \dot{a}_\beta + (P/U) \delta_{\alpha 4} a_4 = -R_{\alpha\beta} b_\beta \tag{4.5}$$

where

$$\|S_{\alpha\beta}\| \equiv \begin{Vmatrix} 2U\delta_{ik} & Q_i \\ Q_i & -2UC \end{Vmatrix} \tag{4.6}$$

and b_α denotes a vector in $\mathscr{V}_4$, which has components $\bar{\bar{v}}_i$ and $\bar{\bar{\theta}}$.

Multiplying the principal vectors $n_\alpha^{(\gamma)}$ ($\gamma = 1, 2, 3, 4$) of $R_{\alpha\beta}$ with (4.5) we have a set of differential equations:

$$T_\alpha^{(\gamma)} \dot{a}_\alpha + (P/U_\gamma) n_4^{(\gamma)} a_4 = 0\,, \tag{4.7}$$

where

$$T_\alpha^{(\gamma)} \equiv S_{\alpha\beta} n_\beta^{(\gamma)} \tag{4.8}$$

and U_γ is the principal velocity corresponding to the principal vector $n_\alpha^{(\gamma)}$. Equations (4.7) govern the amplitudes of the waves. If the material is a non-conductor, we have $P=0$, or if the wave is purely mechanical, we have $n_4^{(i)}=0$. In these cases we have $T_\alpha^{(\gamma)} \dot{a}_\alpha = 0$, which means that, for the case of $\det(T_\alpha^{(\gamma)}) \neq 0$,

$$a_\alpha = \text{constant}\,. \tag{4.9}$$

Then we can say that plane thermo-acoustical waves in a non-conductor and purely mechanical plane waves propagate with constant amplitudes along their propagation paths. However, when $Pn_4^{(\gamma)} \neq 0$, the amplitudes of the wave may vary, in general, along the paths.

With respect to the variation of the wave amplitude there are, in general, three effects: (a) *geometrical effect*, (b) *non-linear effect* and (c) *thermal effect*.

When the wave front is not a plane, e.g., it is a spherical surface, it must spread or shrink according to its centrifugal or centripetal propagation direction, then its amplitude must, in general, decay or grow. This fact denotes the geometrical effect.

When we do not restrict our attention to small deviations of deformation and temperature from an equilibrium state, the constitutive equations may have non-linear terms with respect to them, then the derived differential equations for the wave amplitudes are not (4.7) but they contain second- and higher-order terms of the amplitudes. Therefore the wave amplitudes may decay or grow by reason of existence of the non-linear terms. This is the non-linear effect.

Here we consider *plane* waves in *linear* thermo-elastic materials. Then we have no geometrical and non-linear effects. However, as we analyzed above, plane thermo-acoustical waves in linear thermo-elastic materials may vary their amplitudes along the paths, and the cause of this variation may be regarded as heat conduction or thermal coupling. This is the thermal effect.

5. Linear isotropic thermo-elastic materials

In this case C_{ijkl}, C_{ij}, v_{ij} and K_{ij} are constant isotropic tensors with symmetry relations (2.7). We assume now $v_{ij}=v_{ji}$ and $K_{ij}=K_{ji}$. Then by the familiar theorem of tensor analysis we can express them as

$$C_{ijkl} = (\lambda/\rho_0)\delta_{ij}\delta_{kl}+(\mu/\rho_0)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \tag{5.1a}$$

$$C_{ij} = -(3\lambda+2\mu)\alpha(T_0/\rho_0)\delta_{ij} \equiv -A\delta_{ij}, \tag{5.1b}$$

$$C = -c_V T_0^2, \quad v_{ij} = (1/\tau)\delta_{ij}, \quad K_{ij} = \varkappa T_0 \delta_{ij}, \tag{5.1c, d, e}$$

where λ and μ are the Lamé elastic constants, α is the coefficient of thermal expansion, c_V is the specific heat at constant volume, and δ_{ij} denotes the Kronecker delta.

Substituting (5.1) into (3.5) and (4.4) we have

$$Q_{ik} = ((\lambda+\mu)/\rho_0)n_i n_k+(\mu/\rho_0)\delta_{ik}, \tag{5.2a}$$

$$Q_i = -(3\lambda+2\mu)\alpha(T_0/\rho_0)n_i = -An_i, \tag{5.2b}$$

$$Q = \frac{\varkappa T_0}{\rho_0 \tau}, \quad P = \frac{\varkappa T_0}{\rho_0 \tau^2}. \tag{5.3a, b}$$

Then adopting the coordinate axis $x_3 = z$ as the propagation direction of the plane wave we obtain

$$\|R_{\alpha\beta}\| = \left\| \begin{matrix} c_T^2-U^2 & 0 & 0 & 0 \\ 0 & c_T^2-U^2 & 0 & 0 \\ 0 & 0 & c_L^2-U^2 & AU \\ 0 & 0 & AU & \dfrac{\varkappa T_0}{\rho_0\tau} - c_V T_0^2 U^2 \end{matrix} \right\|, \tag{5.4}$$

$$\|S_{\alpha\beta}\| = \left\| \begin{matrix} 2U & 0 & 0 & 0 \\ 0 & 2U & 0 & 0 \\ 0 & 0 & 2U & -A \\ 0 & 0 & -A & 2c_V T_0^2 U \end{matrix} \right\|, \tag{5.5}$$

where

$$c_T \equiv \left(\frac{\mu}{\rho_0}\right)^{\frac{1}{2}}, \quad c_L \equiv \left(\frac{\lambda+2\mu}{\rho_0}\right)^{\frac{1}{2}} \tag{5.6}$$

denote, respectively, the transverse and longitudinal propagation velocities of the purely mechanical wave.

From (3.8) and (5.4) we have four principal waves. Two *shear waves* are purely mechanical and propagate with same velocity c_T and they have, respectively, amplitudes a_1 and a_2. The other two waves are *thermo-longitudinal waves* whose velocities are solutions of

$$(U/c_L)^4-(1+\beta^2+\gamma)(U/c_L)^2+\beta^2=0\,, \tag{5.7}$$

where

$$\beta^2\equiv\frac{\varkappa}{(\lambda+2\mu)c_V\tau T_0}\,,\quad \gamma\equiv\frac{(3\lambda+2\mu)^2\alpha^2}{\rho_0(\lambda+2\mu)c_V} \tag{5.8}$$

are dimensionless material constants*.

From (5.7) we have

$$(U/c_L)^2-1=\tfrac{1}{2}[(\beta^2+\gamma-1)\pm\{(\beta^2+\gamma-1)^2+4\gamma\}^{\frac{1}{2}}]\,. \tag{5.9}$$

Then we can easily prove that $U_+\geqq c_L$ and $U_-\leqq c_L$, where U_+ and U_- ($U_+\geqq U_-$) are two solutions of (5.7), and we can say that $U_+=c_L$ if, and only if $\beta\leqq 1$ and $\gamma=0$, and that $U_-=c_L$ if, and only if $\beta\geqq 1$ and $\gamma=0$. For a non-conductor, which is characterized by $\beta=0$, we have

$$U_-=0\,,\quad U_+=(1+\gamma)^{\frac{1}{2}}c_L\,, \tag{5.10}$$

and for no thermo-mechanical coupling, which is characterized by $\gamma=0$, we have

$$U=c_L\,,\quad U=\beta c_L\,. \tag{5.11}$$

Figure 1 shows the propagation velocities of the thermo-longitudinal waves.

The amplitude ratio of the thermal and mechanical components of the thermo-longitudinal wave is given by

$$\frac{(3\lambda+2\mu)\alpha T_0 c_L}{\lambda+2\mu}\frac{a_4}{a_3}=\frac{(U/c_L)^2-1}{(U/c_L)}\,. \tag{5.12}$$

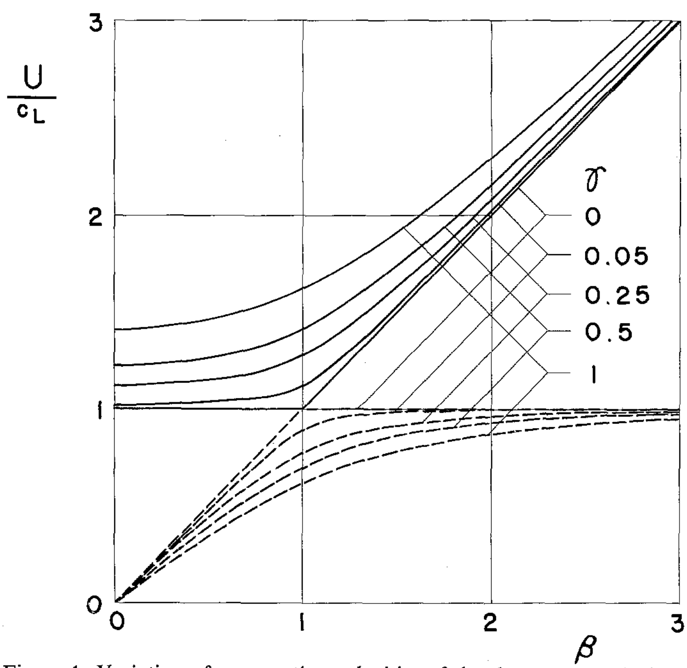

Figure 1. Variation of propagation velocities of the thermo-acoustical waves, where solid and broken lines refer, respectively, to faster and slower waves.

* Equation (5.7) is identical with that of Achenbach [4], while the two dimensionless constants (5.8) are different from those defined by him.

Thus the waves with U_+ ($>c_L$) and U_- ($<c_L$) have, respectively, the same and the opposite sign as the mechanical and thermal amplitudes and the wave with $U=c_L$ is purely mechanical. The variations of the amplitude ratios are shown in Figure 2.

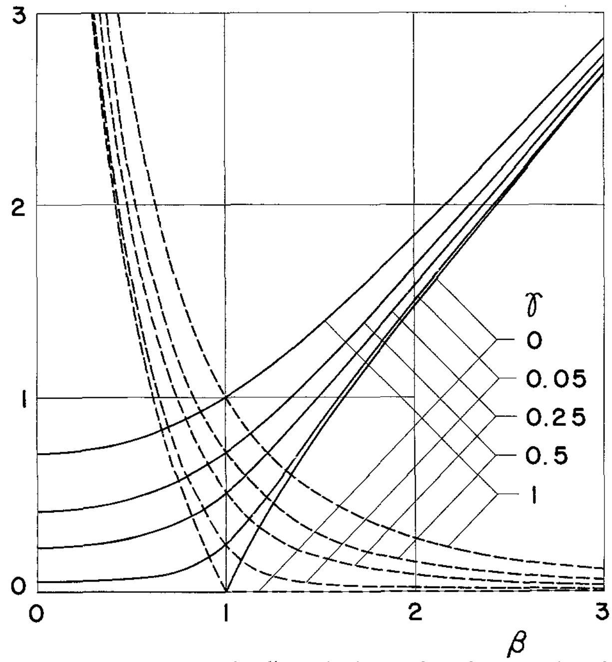

Figure 2. Variation of the dimensionless ratios of amplitudes of the thermo-acoustical waves, where the ordinate denotes $[(3\lambda+2\mu)\alpha T_0 c_L/(\lambda+2\mu)]|a_4/a_3|$ and solid and broken lines refer, respectively, to faster and slower waves.

Four principal directions in $\mathscr{V}_4$ are given by

$$n_\alpha^{(1)}=(1,0,0,0)\,,\quad n_\alpha^{(2)}=(0,1,0,0)\,,\quad n_\alpha^{(\pm)}=(0,0,-AU_\pm\,,\ c_L^2-U_\pm^2)\,. \tag{5.13a, b, c}$$

From (4.7), (4.8), (5.5), $(5.6)_1$ and (5.13 a and b) we have

$$\dot{a}_1=\dot{a}_2=0\,, \tag{5.14}$$

which show that the plane shear waves propagate with constant amplitudes. Also from (5.13c) we can easily obtain

$$A(c_L^2+U^2)\dot{a}_3-U\{A^2+2(c_L^2-U^2)c_V T_0^2\}\dot{a}_4=\frac{P(c_L^2-U^2)}{U}a_4\,, \tag{5.15}$$

which yields from (5.12) that

$$\dot{a}_3=-(\nu/\tau)a_3\,, \tag{5.16}$$

where

$$\nu\equiv\frac{\beta^2}{2}\frac{\{(U/c_L)^2-1\}^2}{(U/c_L)^2[\gamma+\{(U/c_L)^2-1\}^2]} \tag{5.17}$$

denotes the dimensionless damping constant*. Therefore from (5.16) and (5.12) we have

$$a_3=a_3(0)\exp\left(-\frac{\nu}{\tau}t\right)=a_3(0)\exp\left(-\frac{\nu}{U\tau}z\right), \tag{5.18a}$$

$$a_4=a_4(0)\exp\left(-\frac{\nu}{\tau}t\right)=a_4(0)\exp\left(-\frac{\nu}{U\tau}z\right), \tag{5.18b}$$

* The dimensionless damping constant (5.17) is different from the one given by Achenbach [4].

where $a_3(0)$ and $a_4(0)$ denote, respectively, the initial amplitudes of a_3 and a_4 at $t=0$ or $z=0$ and where U denotes either U_+ or U_-.

From (5.17) and (5.18) we can say that thermo-longitudinal waves decay exponentially with respect to time or distance, and their amplitudes remain constant if, and only if the material is a non-conductor or $U_\pm = c_L$. The variation of the damping constant is shown in Figure 3.

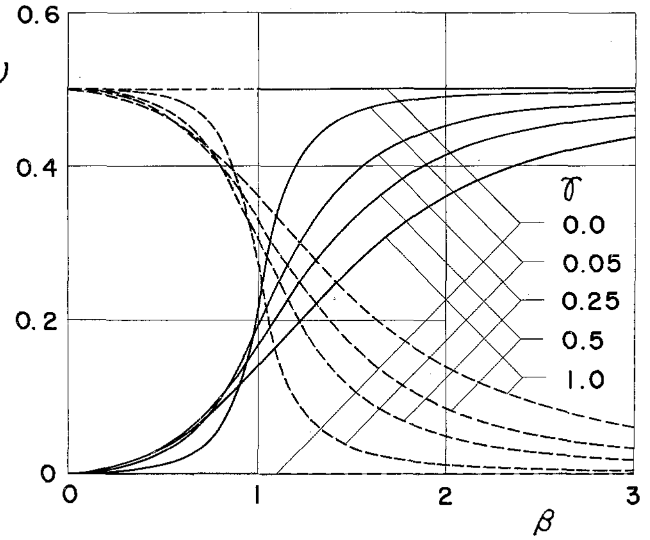

Figure 3. Variation of the dimensionless damping constant, where solid and broken lines refer, respectively, to faster and slower waves.

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